

ON THE SCHRÖDINGER MAXIMAL FUNCTION IN HIGHER DIMENSION

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(DEDICATED TO B. KASIN FOR HIS 60th)

ABSTRACT. New estimates on the maximal function associated to the linear Schrödinger equation are established.

1. Introduction

Recall that the solution of the linear Schrödinger equation

$$\begin{cases} iu_t - \Delta u = 0 \\ u(x, 0) = f(x) \end{cases} \quad (1.0)$$

with $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ is given by

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int e^{i(x \cdot \xi + t|\xi|^2)} \widehat{f}(\xi) d\xi. \quad (1.1)$$

Assuming $f \in H^s(\mathbb{R}^n)$ for suitable s , when does the almost convergence property

$$\lim_{t \rightarrow 0} e^{it\Delta} f = f \text{ a.e.} \quad (1.2)$$

hold? This problem originates from Carleson's paper [C] who proved convergence for $s \geq \frac{1}{4}$ when $n = 1$. Dahlberg and Kenig [D-K] showed that this result is sharp. In dimensions $n \geq 2$, Shölin [S] and Vega [V] established independently convergence for $s > \frac{1}{2}$, while similar examples as considered in [D-K] show failure of convergence for $s < \frac{1}{4}$.

The problem for $n = 2$ has been studied by various authors. Proof of convergence for some $s < \frac{1}{2}$ appears first in the author's papers [B1], [B2]. Subsequently, the required threshold was lowered by Moyua, Vargas, Vega [M-V-V], Tao, Vargas [T-V1,2] and S. Lee [L]. The strongest result to date appears in [L] and asserts a.e. convergence for $f \in H^s(\mathbb{R}^2)$, $s > \frac{3}{8}$. On the other hand, no improvements of Sjölin-Vega result for $n \geq 3$ seem to have been obtained so far.

We will prove the following

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

Theorem 1. *For all n , there is $\varepsilon_n > 0$ such that the a.e. convergence property holds for $f \in H^s(\mathbb{R}^n)$, $s > \frac{1}{2} - \varepsilon_n$.*

In fact, one may take $\varepsilon_n = \frac{1}{4n}$.

Note that for $n = 2$, the exponent corresponds to the one obtained in [L]. The reason for formulating Theorem 1 this way is that the first (qualitative) statement allows for a less involved argument that will be presented first.

Most of the progress on this problem relates to the advances around the restriction theory for the paraboloid. In particular, the method used in [L] depends on Tao's bilinear restriction estimate [T]. The proof of Theorem 1 will be based on the results and techniques from [B-G] and hence ultimately on the multi-linear restriction theory developed in [B-C-T].

As pointed out in [L], it should also be observed that all results obtained in the convergence problem for (1.0) hold equally well for generalized Schrödinger equations

$$iu_t - \Phi(D)u = 0 \tag{1.3}$$

with $\Phi(D)$ a Fourier multiplier operator satisfying $|D^\alpha \phi(\xi)| \leq C|\xi|^{2-\alpha}$, $|\nabla \phi(\xi)| \geq c|\xi|$. This is also the case for Theorem 1.

Perhaps the most interesting point in this Note is a disproof of what one seemed to believe, namely that $f \in H^s(\mathbb{R}^n)$, $s > \frac{1}{4}$, should be the correct condition in arbitrary dimension n .

Theorem 2. *For $n > 4$, the a.e. convergence property for the Schrödinger group requires $f \in H^s(\mathbb{R}^n)$ with $s \geq \frac{n-2}{2n}$.*

Hence this exponent $\rightarrow \frac{1}{2}$ for $n \rightarrow \infty$. The examples are of a quite different nature from previous constructions and rely on an arithmetical input, namely lattice points on spheres.

Returning to the generalized setting, we exhibit in the last section of the paper some further examples setting further restrictions to what may be proven in the generality of our approach to Theorem 1.

2. Braking the $\frac{1}{2}$ -barrier

Before getting to more quantitative analysis, we start by establishing the first part of Theorem 1. The main input in that argument is the multi-linear bound from [B-C-T] and it is technically rather easy.

A few preliminary reductions. Assume $R > 1$ large and

$$\text{supp } \widehat{f} \subset \underset{2}{[|\xi| \sim R]}. \tag{2.0}$$

Our aim is to establish a maximal inequality of the form

$$\left\| \sup_{0 < t < 1} |e^{it\Delta} f| \right\|_{L^2(B_1)} \leq CR^\alpha \|f\|_2 \quad (2.1)$$

where $B_K = [x \in \mathbb{R}^n; |x| \leq K]$. From the stationary phase, (2.1) may be deduced from the restricted inequality

$$\left\| \sup_{0 < t < \frac{1}{R}} |e^{it\Delta} f| \right\|_{L^2(B_1)} \leq CR^{\alpha'} \|f\|_2 \quad (2.2)$$

for any $\alpha' < \alpha$ (see [L], Lemma 2.3). Hence we restrict $0 < t < \frac{1}{R}$ in (2.1).

Next, performing a rescaling, we need to show that

$$\left\| \sup_{0 < t < R} |e^{it\Delta} f| \right\|_{L^2(B_R)} \leq CR^\alpha \|f\|_2 \quad (2.3)$$

assuming

$$\text{supp } \widehat{f} \subset [|\xi| \sim 1]. \quad (2.4)$$

This format corresponds to [B-G]. What follows is closely related to arguments in §2, §3 of that paper.

For notational convenience, set $x_{n+1} = t$ and

$$\psi(x, \xi) = x_1 \xi_1 + \cdots + x_n \xi_n + x_{n+1} |\xi|^2. \quad (2.5)$$

Hence

$$(e^{it\Delta} f)(x_1, \dots, x_n) = \int_{\Omega} \widehat{f}(\xi) e^{i\psi(x, \xi)} d\xi \equiv T\widehat{f} \quad (2.6)$$

with

$$\Omega = [|\xi| \sim 1].$$

Fix $1 \ll K = K(R) \ll R$ to be specified later. Partition Ω in boxes Ω_α of size $\frac{1}{K}$ centered at ξ_α . Then

$$Tf = \sum_{\alpha} e^{i\psi(\alpha, \xi_\alpha)} T_\alpha f \quad (2.7)$$

with

$$T_\alpha f(x) = \int_{\Omega_\alpha} f(\xi) e^{i[\psi(x, \xi) - \psi(x, \xi_\alpha)]} d\xi. \quad (2.8)$$

Note that $|\nabla_x [\psi(x, \xi) - \psi(x, \xi_\alpha)]| \lesssim \frac{1}{K}$ and hence $T_\alpha f$ may be viewed as ‘approximately constant’ on balls of size $O(K)$ in $B_R \subset \mathbb{R}^{n+1}$. We refer the reader to [B-G], §2, for the technical details needed to formalize this last statement.

Fix a ball $B_K \subset B_R$. The main idea is to exploit a dichotomy similar to the one used in §2, §3 of [B-G]. More precisely, we distinguish the following two alternatives.

Case I: ‘ $(n-1)$ -transversality’

Viewing the $T_\alpha f$ as essentially constant on Ω_α , this means that there are indices $\alpha_1, \dots, \alpha_{n+1}$ such that

$$(2.9) \quad |\nu(\xi'_1) \wedge \dots \wedge \nu(\xi'_{n+1})| > K^{-C} \text{ whenever } \xi'_j \in \Omega_{\alpha_j}.$$

Here $\nu(\xi) // -2 \sum_{i=1}^n \xi_i e_i + e_{n+1}$ denotes the normal of the hypersurface $(\xi, |\xi|^2)$

and

$$(2.10) \quad |T_{\alpha_1} f|, \dots, |T_{\alpha_{n+1}} f| \geq K^{-n} \max_\alpha |T_\alpha f|.$$

Hence the following majoration holds on B_K

$$|Tf| \lesssim K^n \max_\alpha |T_\alpha f| \lesssim K^{2n} \left[\prod_{j=1}^{n+1} |T_{\alpha_j} f| \right]^{\frac{1}{n+1}} \leq K^{2n} \sum_{\substack{\alpha_1, \dots, \alpha_{n+1} \\ \text{transversal}}} \left[\prod_{j=1}^{n+1} |T_{\alpha_j} f| \right]^{\frac{1}{n+1}}.$$

Recall the multi-linear restriction bound from [B-C-T], implying that for $\alpha_1, \dots, \alpha_{n+1}$ ‘transversal’ as defined in (2.9)

$$\left\| \left[\prod_{j=1}^{n+1} |T_{\alpha_j} f| \right]^{\frac{1}{n+1}} \right\|_{L^q(B_R)} \ll K^C R^\varepsilon \|f\|_2 \quad (2.12)$$

holds, with $q = \frac{2(n+1)}{n}$.

Hence, denoting $x = (x', x_{n+1})$ and using Hölder’s inequality, the corresponding contribution to the maximal function may be bounded by

$$\begin{aligned} & \left\| \sup_{|x_{n+1}| < R} (2.11) \right\|_{L^2_{[|x'| < R]}} \lesssim \\ & \left\| (2.11) \right\|_{L^2_{[|x'| < R]}} L^q_{[|x_{n+1}| < R]} \lesssim \\ & R^{n(\frac{1}{2} - \frac{1}{q})} \left\| (2.11) \right\|_{L^q_{[|x| < R]}} \stackrel{(2.12)}{\ll} \\ & R^{n(\frac{1}{2} - \frac{1}{q}) + \varepsilon} K^C \|f\|_2 = R^{\frac{n}{2(n+1)} + \varepsilon} K^C \|f\|_2. \end{aligned} \quad (2.13)$$

Case II: Failure of $(n+1)$ -transversality

In this situation, there is an $(n-1)$ -dimensional affine hyperplane \mathcal{L} in \mathbb{R}^n such that $\text{dist}(\Omega_\alpha, \mathcal{L}) \lesssim \frac{1}{K}$ if $|T_\alpha f| \geq K^{-n} \max_\alpha |T_\alpha f|$ on B_K .

Denoting $\tilde{\mathcal{L}}$ and $\frac{1}{K}$ -neighborhood of \mathcal{L} , it follows that on B_K

$$|Tf(x)| \leq \left| \sum_{\Omega_\alpha \subset \tilde{\mathcal{L}}} e^{i\psi(x, \xi_\alpha)} T_\alpha f(x) \right| + \max_\alpha |T_\alpha f(x)| \quad (2.14)$$

and we write on B_K

$$\sum_{\Omega_\alpha \subset \tilde{\mathcal{L}}} e^{i\psi(x, \xi_\alpha)} T_\alpha f(x) = \phi_{B_K}(x) \cdot \left(\sum_{\Omega_\alpha \subset \tilde{\mathcal{L}}} |T_\alpha f|^2 \right)^{1/2}. \quad (2.15)$$

Next, define a function ϕ by

$$\phi|_{B_K} = \phi_{B_K}$$

for B_K satisfying alternative II. It follows from (2.14) that the corresponding contribution to the maximal function is bounded by

$$\left\| \max_{|x_{n+1}| < R} \left\{ \phi \left(\sum_\alpha |T_\alpha f|^2 \right)^{\frac{1}{2}} \right\} \right\|_{L^2[|x'| < R]}. \quad (2.16)$$

Let $\{B_{\gamma, \ell}\}$, $B_{\gamma, \ell} = B_\gamma \times I_\ell \subset \mathbb{R}^n \times \mathbb{R}$ be a partition of $B(0, R)$ in K -cubes. Clearly

$$(2.16) \leq \left\{ \sum_{\gamma, \ell} \left(\sum_\alpha |T_\alpha f|^2 \right) \Big|_{B_{\gamma, \ell}} \left[\int_{B_\gamma} \max_{x_{n+1} \in I_\ell} |\phi_{\gamma, \ell}|^2 dx' \right] \right\}^{1/2}. \quad (2.17)$$

Assume we dispose over a bound

$$\left[\int_{B_\gamma} \max_{x_{n+1} \in I_\ell} |\phi_{\gamma, \ell}|^2 dx' \right]^{1/2} < A \quad (2.18)$$

where \int_{B_γ} stands for $\frac{1}{\text{mes } B_\gamma} \int_{B_\gamma}$.

Then (2.17) is bounded by

$$\begin{aligned} & A \left\{ \frac{1}{K} \sum_{\gamma, \ell} \int_{B_{\gamma, \ell}} \left(\sum_\alpha |T_\alpha f|^2 \right) dx \right\}^{1/2} \\ &= A \left(\frac{1}{K} \right)^{\frac{1}{2}} \left\| \left(\sum_\alpha |T_\alpha f|^2 \right)^{1/2} \right\|_{L^2(|x| < R)} \\ &\leq CA \left(\frac{R}{K} \right)^{1/2} \left(\sum_\alpha \|f|_{\Omega_\alpha}\|_2^2 \right)^{1/2} \\ &= CA \left(\frac{R}{K} \right)^{1/2} \|f\|_2. \end{aligned} \quad (2.19)$$

It remains to establish a bound on A in (2.18).

Again, since in (2.15) the $T_\alpha f$ are viewed as constant on B_K , we need a bound of the form

$$\left\| \max_{|x_{n+1}| < K} \left| \sum_{\alpha} a_{\alpha} e^{i\psi(x, \xi_{\alpha})} \right| \right\|_{L^2[|x'| < K]} \leq AK^{\frac{n}{2}} \left(\sum |a_{\alpha}|^2 \right)^{1/2} \quad (2.20)$$

where the points $\{\xi_{\alpha}\}, |\xi_{\alpha}| \sim 1$, are $\frac{1}{K}$ -separated and in an $(n-1)$ -dim affine hyperplane $\mathcal{L} \subset \mathbb{R}^n$.

Performing a rotation around the e_{n+1} -axis, we may assume $\xi_{\alpha} = c = \text{constant}$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{L}$. Hence the left side of (2.20) becomes

$$K^{\frac{1}{2}} \left\| \max_{|x_{n+1}| < K} \left| \sum_{\alpha} a_{\alpha} e^{i(x_1 \xi_{\alpha,1} + \dots + x_{n-1} \xi_{\alpha,n-1} + x_{n+1}(\xi_{\alpha,1}^2 + \dots + \xi_{\alpha,n-1}^2))} \right| \right\|_{L^2[|x_1|, \dots, |x_{n-1}| < K]} \quad (2.21)$$

At this point, the dimension is reduced from n to $n-1$.

Let $\{\Omega'_{\alpha}\}$ be disjoint $\frac{1}{K}$ -neighborhoods of the points $\{(\xi_{\alpha,1}, \dots, \xi_{\alpha,n-1})\}$ in \mathbb{R}^{n-1} .

Let g on $[|\xi'| \sim 1] \subset \mathbb{R}^{n-1}$ be defined by

$$g(\xi') = \begin{cases} a_{\alpha} |\Omega'_{\alpha}|^{-1} & \text{if } \xi' \in \Omega'_{\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\|g\|_2 \sim K^{\frac{n-1}{2}} \left(\sum |a_{\alpha}|^2 \right)^{1/2} \quad (2.22)$$

and

$$(2.21) \sim K^{\frac{1}{2}} \left\| \max_{|t| < K} \left| \int g(\xi') e^{i(y \cdot \xi' + t|\xi'|^2)} \right| \right\|_{L^2[|y| < K]}. \quad (2.23)$$

Denote θ_{n-1} an exponent for which (2.3) holds in dimension $n-1$.

From (2.22), it follows that

$$(2.23) \lesssim K^{\frac{1}{2} + \theta_{n-1}} K^{\frac{n-1}{2}} \left(\sum_{\alpha} |a_{\alpha}|^2 \right)^{1/2}$$

which shows that we may take $A \sim K^{\theta_{n-1}}$. Hence in (2.19), we obtain the estimate

$$CR^{\frac{1}{2}} K^{\theta_{n-1} - \frac{1}{2}} \|f\|_2. \quad (2.24)$$

Together with the Case I contribution (2.13), we obtain the bound

$$\left\| \sup_{0 < t < R} |e^{it\Delta} f| \right\|_{L^2(B_R)} \ll [R^{\frac{n}{2(n+1)} + \varepsilon} K^C + R^{\frac{1}{2}} K^{\theta_{n-1} - \frac{1}{2}}] \|f\|_2 \quad (2.25)$$

where K is a parameter. Recall that $\theta_{n-1} < \frac{1}{2}$. Hence an appropriate choice of $K = R^\delta$ permits to obtain (2.3) for some $\theta = \theta_n < \frac{1}{2}$.

3. A quantitative estimate

The proof of the second part of Theorem 1 depends on the more refined analysis from §4 in [B-G] and its higher dimensional version.

Before stating the relevant result from [B-G], we fix some terminology. Let

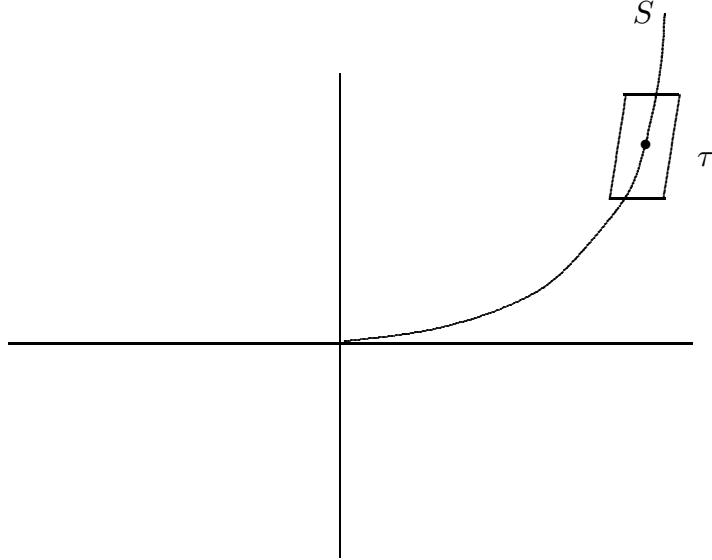
$$S = \{(\xi, |\xi|^2), |\xi| \sim 1\} \subset \mathbb{R}^{n+1}.$$

For fixed $\delta > 0$, consider a partition of $\{\xi : |\xi| \sim 1\} \subset \mathbb{R}^n$ in cells Q of size δ and denote

$$\tau = \{(\xi, |\xi|^2); \xi \in Q\} \tag{3.1}$$

the corresponding partition of S in δ -caps. Thus the convex hull $\text{conv}(\tau)$ is a $(\underbrace{\delta \times \cdots \times \delta}_n \times \delta^2)$ -box tangent to S and we denote $\overset{\circ}{\tau} \subset \mathbb{R}^{n+1}$ the polar box of $\text{conv}(\tau)$

shifted with 0 as center. Thus $\overset{\circ}{\tau}$ is a $(\frac{1}{\delta} \times \cdots \times \frac{1}{\delta} \times \frac{1}{\delta^2})$ -box.



Denote also

$$f_\tau = f|_Q$$

with Q and τ related by (3.1).

Let the operator T be as in previous section.

Next recall (3.4) - (3.8) from §4 in [B-G], providing an estimate for Tf on B_R in $3D$ (i.e. $n = 2$). Thus

$$\begin{aligned}
|Tf| &\ll \\
R^\varepsilon \max_{\frac{1}{\sqrt{R}} < \delta < 1} \max_{\mathcal{E}_\delta} &\left[\sum_{\tau \in \mathcal{E}_\delta} (\phi_\tau |Tf_{\tau_1}|^{1/3} |Tf_{\tau_2}|^{1/3} |Tf_{\tau_3}|^{1/3})^2 \right]^{1/2} \\
R^\varepsilon \max_{\mathcal{E}_{\frac{1}{\sqrt{R}}}} &\left[\sum_{\tau \in \mathcal{E}} (\phi_\tau |Tf_\tau|)^2 \right]^{1/2}
\end{aligned} \tag{3.2}$$

where

(3.3) \mathcal{E}_δ consists of at most $\frac{1}{\delta}$ disjoint δ -caps.

(2.4) $\tau_1, \tau_2, \tau_3 \subset \tau$ are 3-transversal $\frac{\delta}{K}$ -caps (K is a large constant).

(3.5) For each τ , $\phi_\tau \geq 0$ is a function on \mathbb{R}^n satisfying

$$\int_B \phi_\tau^4 \ll R^\varepsilon$$

for all B taken in a tiling of \mathbb{R}^{n+1} with translates of τ .

Of course (3.2) certainly implies

$$|Tf| \ll R^\varepsilon \sum_{\substack{\delta \text{ dyadic} \\ \frac{1}{\sqrt{R}} < \delta < 1}} \left[\sum_{\tau \text{ } \delta\text{-cap}} (\phi_\tau |Tf_{\tau_1}|^{1/3} |Tf_{\tau_2}|^{1/3} |Tf_{\tau_3}|^{1/3})^2 \right]^{1/2} \tag{3.6}$$

$$+ R^\varepsilon \left[\sum_{\tau \text{ } \frac{1}{\sqrt{R}}\text{-cap}} (\phi_\tau |Tf_\tau|)^2 \right]^{1/2} \tag{3.7}$$

where $\sum_{\tau \text{ } \delta\text{-cap}}$ refers to summation over a partition of S in δ -caps τ .

Formula (3.2) has a higher dimensional version, deduced in a similar way from §3 in [B-G]. In particular, we get for general $n \geq 2$

$$|Tf| \ll R^\varepsilon \sum_{\delta} \left[\sum_{\tau \text{ } \delta\text{-cap}} \left(\phi_\tau \prod_{j=1}^{n+1} |Tf_{\tau_j}|^{\frac{1}{n+1}} \right)^2 \right]^{\frac{1}{2}} \tag{3.8}$$

$$+ R^\varepsilon \left[\sum_{\tau \text{ } \frac{1}{\sqrt{R}}\text{-cap}} (\phi_\tau |Tf_\tau|)^2 \right]^{\frac{1}{2}} \tag{3.9}$$

where now $\tau_1, \dots, \tau_{n+1} \subset \tau$ are $(n+1)$ -transversal $\frac{\delta}{K}$ -caps and $\phi_\tau \geq 0$ on \mathbb{R}^{n+1} satisfies

$$\int_B \phi_\tau^q \ll R^\varepsilon \text{ with } q = \frac{2n}{n-1} \quad (3.10)$$

for all B in a tiling of \mathbb{R}^{n+1} from $\overset{\circ}{\tau}$ -translates.

It is now clear from (3.8), (3.9) and convexity that in order to bound

$$\|Tf\|_{L^2_{[|x'| < R]} L^\infty_{[|x_{n+1}| < R]}}$$

it suffices to estimate

$$\left\| \phi_\tau \left(\prod_{j=1}^{n+1} |Tf_{\tau_j}|^{\frac{1}{n+1}} \right) \right\|_{L^2_{[|x'| < R]} L^\infty_{[|x_{n+1}| < R]}} \quad (3.11)$$

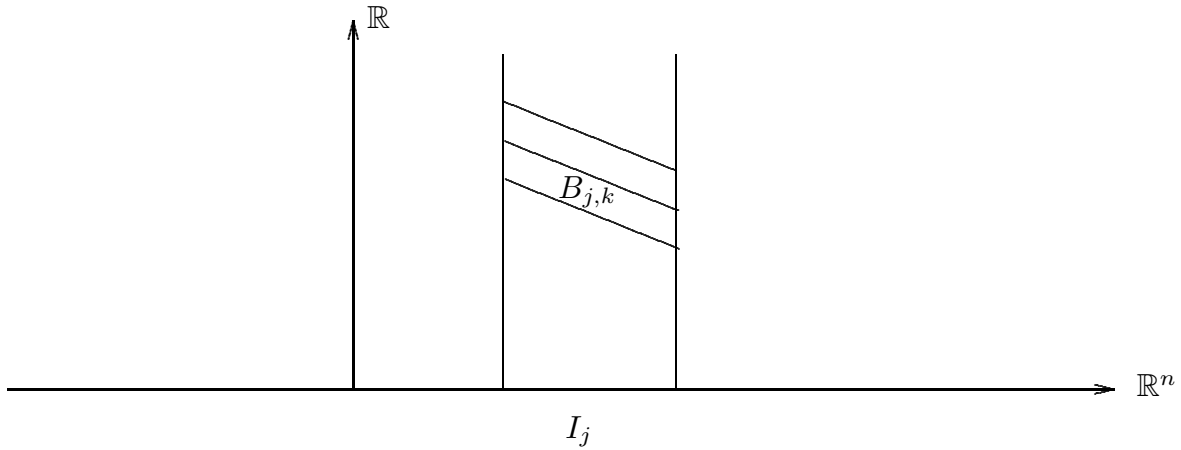
with $\tau_1, \dots, \tau_{n+1} \subset \tau$ as above, and also

$$\|\phi_\tau |Tf_\tau| \|_{L^2_{[|x'| < R]} L^\infty_{[|x_{n+1}| < R]}} \quad (3.12)$$

with τ or $\frac{1}{\sqrt{R}}$ -cap.

Denote $\phi = \phi_\tau$ and $F = \prod_{j=1}^{n+1} |Tf_{\tau_j}|^{\frac{1}{n+1}}$.

Let $\{B_{jk}\}$ be a tiling of $B(0, R) \subset \mathbb{R}^{n+1}$ with $\pi_{x'}(B_{jk}) = I_j \subset \mathbb{R}^n$ a partition of $B(0, R) \subset \mathbb{R}^n$ in $\underbrace{\left(\frac{1}{\delta} \times \dots \times \frac{1}{\delta} \right)}_{n-1} \times \frac{1}{\delta^2}$ -boxes and $B_{jk} \sim \text{translate of } \overset{\circ}{\tau}$



Let

$$p = \frac{2(n+1)}{n} \quad \text{and} \quad q = \frac{2n}{n-1}.$$

By Hölder's inequality

$$\begin{aligned} (3.11) &= \left[\sum_j \|\phi F\|_{L_{I_j}^2 L_{[|x_{n+1}| < R]}^\infty}^2 \right]^{\frac{1}{2}} \\ &\leq \delta^{-\frac{1}{2}} \left[\sum_j \|\phi F\|_{L_{I_j}^p L_{x_{n+1}}^\infty}^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (3.13)$$

Writing for fixed $x' \in I_j$

$$\sup_{x_{n+1}} |\phi F|(x', x_{n+1}) \leq \left(\sum_k \|\phi F\|_{L^\infty(B_{j,k}(x'))}^p \right)^{\frac{1}{p}}$$

it follows that

$$(3.13) \leq \delta^{-\frac{1}{2}} \left[\sum_j \left(\sum_k \|\phi F\|_{L_{x'}^p L_{x_{n+1}}^\infty(B_{j,k})}^p \right)^{\frac{2}{p}} \right]^{\frac{1}{2}}. \quad (3.14)$$

Since $\text{supp } \widehat{Tf_{\tau_j}} \subset \tau$, we may view F as essentially constant on each B_{jk} . From (3.10)

$$\begin{aligned} \|\phi\|_{L_{x'}^p L_{x_{n+1}}^\infty(B_{jk})} &\leq \|\phi\|_{L_{x'}^p L_{x_{n+1}}^q(B_{jk})} \\ &\leq |I_j|^{\frac{1}{p} - \frac{1}{q}} \|\phi\|_{L^q(B_{jk})} \\ &\ll R^\varepsilon |I_j|^{\frac{1}{p} - \frac{1}{q}} |B_{jk}|^{\frac{1}{q}} \\ &\ll R^\varepsilon \left(\frac{1}{\delta} \right)^{\frac{n}{2} + \frac{n-1}{2n}} \end{aligned}$$

and hence

$$\begin{aligned} \|\phi F\|_{L_{x'}^p L_{x_{n+1}}^\infty(B_{jk})} &\ll R^\varepsilon \left(\frac{1}{\delta} \right)^{\frac{n}{2} + \frac{n-1}{2n}} |I_j|^{-\frac{1}{2}} \delta^{\frac{1}{p}} \|F\|_{L_{x'}^2 L_{x_{n+1}}^p(B_{jk})} \\ &\ll R^\varepsilon \delta^{\frac{1}{2} + \frac{1}{2n} - \frac{1}{2(n+1)}} \|F\|_{L_{x'}^2 L_{x_{n+1}}^p(B_{jk})}. \end{aligned} \quad (3.15)$$

Substitution of (3.15) in (3.14) gives

$$\begin{aligned} &R^\varepsilon \delta^{\frac{1}{2n} - \frac{1}{2(n+1)}} \left[\sum_j \left(\sum_k \|F\|_{L_{x'}^2 L_{x_{n+1}}^p(B_{jk})}^p \right)^{\frac{2}{p}} \right]^{\frac{1}{2}} \\ &\leq R^\varepsilon \delta^{\frac{1}{2n} - \frac{1}{2(n+1)}} \|F\|_{L_{[|x'| < R]}^2 L_{[|x_{n+1}| < R]}^p}^2. \end{aligned} \quad (3.16)$$

Next, we evaluate $\|F\|_{L_{x'}^2 L_{x_{n+1}}^p}$. Recall the definition of F .

Let ξ_0 be the center of $Q = \pi_\xi(\tau) \subset \mathbb{R}^n$ and write $\xi - \xi_0 = \delta\zeta, |\zeta| = O(1)$, for $\xi \in Q$. Hence, rescaling

$$\begin{aligned} |(Tf_{\tau_j})(x)| &= \delta^n \left| \int e^{i((\delta x' + 2\delta\xi_0 x_{n+1}) \cdot \zeta + \delta^2 |\zeta|^2 x_{n+1})} f_{\tau_j}(\xi_0 + \delta\zeta) d\zeta \right| \\ &= \delta^n \left| \int e^{i((y' + 2\delta^{-1} y_{n+1} \xi_0) \cdot \zeta + |\zeta|^2 y_{n+1})} f_{\tau_j}(\xi_0 + \delta\zeta) d\zeta \right| \\ &= \delta^{\frac{n}{2}} |Tg_{\tilde{\tau}_j}|(y' + 2\delta^{-1} y_{n+1} \xi_0, y_{n+1}) \end{aligned} \quad (3.17)$$

where $(y_1, \dots, y_{n+1}) = (\delta x_1, \dots, \delta x_n, \delta^2 x_{n+1})$ satisfies

$$|y'| < \delta R, |y_{n+1}| < \delta^2 R$$

and $\|g_{\tilde{\tau}_j}\|_2 = 1$.

Denote

$$G = \prod_1^{n+1} |Tg_{\tilde{\tau}_j}|^{\frac{1}{n+1}}$$

where $\tilde{\tau}_1, \dots, \tilde{\tau}_{n+1}$ are transversal $O(1)$ -caps.

Thus

$$\|F\|_{L_{x'}^2 L_{x_{n+1}}^p} = \delta^{-\frac{2}{p}} \|G(y' + 2\delta^{-1} y_{n+1} \xi_0, y_{n+1})\|_{L_{y'}^2 L_{y_{n+1}}^p}. \quad (3.18)$$

Assume (as we may by rotation in $[e_1, \dots, e_n]$ -space) that ξ_0/e_1 .

Estimate using again Hölder's inequality

$$\|G(y_1 + 2\delta^{-1} y_{n+1} \xi_0, y_2, \dots, y_n, y_{n+1})\|_{L_{y'}^2 L_{y_{n+1}}^p} \leq (\delta R)^{\frac{1}{2} - \frac{1}{p}} \|G\|_{L_{y_2, \dots, y_n}^2 L_{y_1, y_{n+1}}^p}.$$

This gives by (3.18)

$$(3.16) < \delta^{\frac{1}{2n} - \frac{n}{n+1}} R^{\frac{1}{2(n+1)} + \varepsilon} \|G\|_{L_{y_2, \dots, y_n}^2 L_{y_1, y_{n+1}}^p} \quad (3.19)$$

where $|y_1|, \dots, |y_n| < \delta R, |y_{n+1}| < \delta^2 R$.

Partition $[|y'| < \delta R]$ in cubes Ω_s of size $\delta^2 R$. Clearly

$$\|G\|_{L_{y_2, \dots, y_n}^2 L_{y_1, y_{n+1}}^p} \leq (\delta^2 R)^{(n-1)(\frac{1}{2} - \frac{1}{p})} \left[\sum_s \|G\|_{L_{y' \in \Omega_s}^p L_{y_{n+1}}^p}^2 \right]^{\frac{1}{2}}. \quad (3.20)$$

Recalling the definition of G and using again the [B-C-T] multi-linear restriction inequality (together with a standard localization argument in y' -space), it follows that

$$\left[\sum_s \|G\|_{L_{y' \in \Omega_s}^p L_{y_{n+1}}^p}^2 \right]^{\frac{1}{2}} \ll R^\varepsilon.$$

Hence

$$(3.16) < \delta^{\frac{1}{2n} - \frac{1}{n+1}} R^{\frac{n}{2(n+1)} + \varepsilon} \quad (3.21)$$

and since $\delta > \frac{1}{\sqrt{R}}$, this implies that

$$(3.11) \ll R^{\frac{1}{2} - \frac{1}{4n} + \varepsilon}. \quad (3.22)$$

Next, consider (3.12) for which we repeat the preceding with $\delta = \frac{1}{\sqrt{R}}$ up to (3.20), with $G = |Tg_{\tilde{\tau}}|$, $\|g_{\tilde{\tau}}\|_2 = 1$. The cubes Ω_s and y_{n+1} are size $O(1)$. Hence the RH -side of (3.20) becomes

$$\|G\|_{L_{[|y'| < \sqrt{R}]}^2 L_{[|y_{n+1}| < O(1)]}^2} \lesssim \|g_{\tilde{\tau}}\|_2 = 1$$

and (3.12) is also bounded by (3.22).

This concludes the proof of Theorem 1.

4. A construction in higher dimension

We prove Theorem 2.

Let $S = \{(\xi, \frac{1}{R}|\xi|^2); \xi \in \mathbb{R}^n, |\xi| \sim R\}$ with R large and let $H \subset \mathbb{R}^{n+1}$ be the hyperplane $[z_1 - z_{n+1}]$. Hence $\pi_\xi(S \cap H) \subset \mathbb{R}^n$ is the $(n-1)$ -sphere

$$\left(\xi_1 - \frac{R}{2}\right)^2 + \xi_2^2 + \cdots + \xi_n^2 = \frac{R^2}{4}. \quad (4.1)$$

Consider the lattice points

$$\mathcal{E} = \mathbb{Z}^n \cap R^{\frac{1}{n}} S^{(n-1)} \quad (4.2)$$

where $S^{(n-1)}$ denotes the unit sphere in \mathbb{R}^n . If $U \in O(n)$ is an arbitrary orthogonal transformation, we have

$$\frac{1}{2} R^{\frac{n-1}{n}} U(\mathcal{E}) \subset \frac{R}{2} S^{(n-1)}$$

and therefore

$$\mathcal{E}_1 = \frac{R}{2} e_1 + \frac{1}{2} R^{\frac{n-1}{n}} U(\mathcal{E}) \subset \pi_\xi(S \cap H). \quad (4.3)$$

Hence $(\xi, \frac{1}{R}|\xi|^2) \in S \cap H$ for $\xi \in \mathcal{E}_1$. Note also that the points of \mathcal{E}_1 are $\sim R^{\frac{n-1}{n}}$ -separated.

Define next

$$\widehat{f} = \frac{1}{|\mathcal{E}_1|^{\frac{1}{2}}} \sum_{\xi \in \mathcal{E}_1} 1_{B(\xi, \rho)} \quad (4.4)$$

with ρ a sufficiently small constant. Hence $\|f\|_2 = O(1)$ and $\text{supp } \widehat{f} \subset B_R$.

Let $|x|, |t| < C$. By construction and taking ρ small enough

$$(e^{i\frac{t}{R}\Delta} f)(x) = \frac{1}{|\mathcal{E}|^{\frac{1}{2}}} \sum_{\xi \in \mathcal{E}_1} \left[\int_{B(0, \rho)} e^{i(x \cdot \zeta + \frac{t}{R} |\xi + \zeta|^2)} d\zeta \right] e^{ix \cdot \xi}$$

and

$$|(e^{i\frac{t}{R}\Delta} f)(x)| \sim |\mathcal{E}|^{\frac{1}{2}} \quad (4.5)$$

provided (x, t) satisfies moreover

$$x \cdot \xi + \frac{t}{R} |\xi|^2 \in \frac{R}{2} x_1 + 2\pi\mathbb{Z} + B\left(0, \frac{1}{10}\right) \text{ for all } \xi \in \mathcal{E}_1. \quad (4.6)$$

Denote $\nu = \frac{1}{\sqrt{2}}(e_1 - e_{n+1})$ the normal of H . Since $(\xi, \frac{1}{R} |\xi|^2) \in H$, it follows that if (x, t) satisfies (4.6), then so does $(x, t) + \mathbb{R}\nu$.

Also, by (4.3)

$$x \cdot \xi = \frac{R}{2} x_1 + \frac{1}{2} R^{\frac{n-1}{n}} U \xi' \cdot x$$

for some $\xi' \in \mathcal{E} \subset \mathbb{Z}^n$. Hence $(x, 0)$ will satisfy (4.6) if x belongs to the dual lattice $\mathcal{L}^* = 4\pi R^{-\frac{n-1}{n}} U(\mathbb{Z}^n)$ of $\mathcal{L} = \frac{1}{4\pi} R^{-\frac{n-1}{n}} U(\mathbb{Z}^n)$. It follows that (4.6) is valid for

$$(x, t) \in (\mathcal{L}^* \times \{0\}) + \mathbb{R}\nu + B\left(0, \frac{1}{100R}\right). \quad (4.7)$$

Our aim is to choose U as to ensure that

$$\pi_x((\mathcal{L}^* \times \{0\}) + [-C, C]\nu) = \mathcal{L}^* + \left[-\frac{C}{\sqrt{2}}, \frac{C}{\sqrt{2}}\right] e_1 \quad (4.8)$$

is $\frac{1}{100R}$ -dense in $B_1 \subset \mathbb{R}^n$. Recalling the definition of \mathcal{L}^* , the issue is to obtain a unit vector $\theta \in \mathbb{R}^n$ so that

$$R^{-\frac{n-1}{n}} \mathbb{Z}^n + [-C, C]\theta \quad (4.10)$$

is $10^{-4}R^{-1}$ -dense. Equivalently, given $x \in \mathbb{R}^n$, there is $\lambda \in \mathbb{R}$, $|\lambda| < O(R^{\frac{n-1}{n}})$ such that

$$\max_{1 \leq j \leq n} \|x_j - \lambda \theta_j\| < 10^{-4} R^{-\frac{1}{n}}. \quad (4.11)$$

We claim that there is $\theta \in \mathbb{R}^n, |\theta| = 1$ satisfying this property (in fact, the typical θ will do). For completeness sake, we will include a proof of this (standard) fact at the end of this section.

The conclusion of the above is that

$$\{x; \max_{|t| < C} |(e^{i\frac{t}{R}\Delta} f)(x)| \gtrsim |\mathcal{E}|^{\frac{1}{2}}\} \supset B_1. \quad (4.12)$$

Remains to obtain a lower bound on $|\mathcal{E}|$. Assume $R^{\frac{1}{n}} \in \mathbb{Z}$. If $n \geq 4$, it is well known that for $L^2 \in \mathbb{Z}_+, L \rightarrow \infty$

$$|\mathbb{Z}^n \cap LS^{(n-1)}| \gtrsim L^{n-2}. \quad (4.13)$$

Thus we may ensure that

$$|\mathcal{E}| \gtrsim R^{\frac{n-2}{n}} \quad (4.14)$$

and the implication of (4.12) is that the almost convergence property may only hold for $s \geq \frac{n-2}{2n}$ for $n \geq 4$. This establishes Theorem 2.

Proof of the claim

Denote $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ the n -dim torus. Let $0 \leq \varphi \leq 1$ on \mathbb{T}^n be a smooth function satisfying

$$\begin{cases} \varphi(y) = 1 & \text{if } \|y\| < \frac{R^{-\frac{1}{n}}}{2 \cdot 10^4} \\ \varphi(y) = 0 & \text{if } \|y\| > \frac{R^{-\frac{1}{n}}}{10^4} \end{cases} \quad (4.15)$$

and with Fourier expansion

$$\varphi(y) = \sum_{k \in \mathbb{Z}^n} \hat{\varphi}(k) e(ky)$$

satisfying

$$|\hat{\varphi}(k)| \leq C_1 R^{-1} (1 + R^{-\frac{1}{n}} |k|)^{-n}. \quad (4.16)$$

To prove (4.11), we show that given any $a \in \mathbb{T}^n$, there is $\lambda \in \mathbb{R}, |\lambda| < O(R^{\frac{n-1}{n}})$ s.t

$$0 < \varphi(a + \lambda\theta) = \hat{\varphi}(0) + \sum_{k \neq 0} \hat{\varphi}(k) e(k.a) e(\lambda k.\theta). \quad (4.17)$$

Proceed by averaging over λ as above. The contribution of the last term in (4.17) may clearly be bounded by

$$\sum_{k \neq 0} |\hat{\varphi}(k)| \left| \int e(\lambda k.\theta) \eta(\lambda) d\lambda \right| \stackrel{(4.16)}{<} C_1 R^{-1} \sum_{k \neq 0} (1 + R^{-\frac{1}{n}} |k|)^{-n} |\hat{\eta}(k.\theta)| \quad (4.18)$$

where we let $\eta(u) = R^{-\frac{n-1}{n}}\eta_1(R^{-\frac{n-1}{n}}u)$ with $0 \leq \eta_1 \leq 1$, $\int_{\mathbb{R}} \eta_1 = 1$ a smooth compactly supported bumpfunction. We may ensure that for $|u| \rightarrow \infty$

$$|\hat{\eta}_1(u)| \leq (1 + C_3|u|)^{-2} \quad (4.19)$$

so that

$$(4.18) < C_1 R^{-1} \sum_{k \neq 0} (1 + R^{-\frac{1}{n}}|k|)^{-n} (1 + C_3 R^{\frac{n-1}{n}}|k \cdot \theta|)^{-2}. \quad (4.20)$$

It will therefore suffice to obtain θ satisfying $(4.20) < \frac{1}{2}\hat{\varphi}(0)$, i.e.

$$\sum_{k \neq 0} (1 + R^{-\frac{1}{n}}|k|)^{-n} (1 + C_3 R^{\frac{n-1}{n}}|k \cdot \theta|)^{-2} < \frac{C(n)}{C_1} \quad (4.21)$$

with $c(n)$ a constant depending on n . This last statement is easily established by integration over θ and taking C_3 large enough.

5. Maximal functions associated to oscillatory integral operators

Returning to Theorem 1, its proof extends to bounding certain maximal functions associated to oscillatory integral operators.

More specifically, assume $\varphi : \Omega \rightarrow \mathbb{R}$ a smooth function defined on a neighborhood Ω of $0 \in \mathbb{R}^n$ of the form

$$\varphi(\xi) = \langle A\xi, \xi \rangle + O(|\xi|^3) \quad (5.1)$$

with A positive definite.

Define

$$(T_R^* f)(x') = \sup_{|x_{n+1}| \leq R} \left| \int_{\Omega} f(\xi) e^{i(x' \cdot \xi + x_{n+1} \varphi(\xi))} d\xi \right| \quad (5.2)$$

with $f \in L^2(\Omega)$, $x' = (x_1, \dots, x_n)$, $|x'| \leq R$.

We proved that

$$\|T_R^* f\|_{L^2_{[|x'| \leq R]}} \ll R^{\frac{1}{2} - \frac{1}{4n} + \varepsilon} \|f\|_2 \quad (5.3)$$

and may ask for the best constant $B(R)$ s.t.

$$\|T_R^* f\|_{L^2_{[|x'| \leq R]}} \leq B(R) \|f\|_2 \quad (5.4)$$

holds in this generality.

Equivalently, let S be a compact, smooth n -dim hypersurface in \mathbb{R}^{n+1} of positive curvature. Let σ be its surface measure. Fix a point $p \in S$; let $p + H$ be the tangent

hyperplane and τ the normal at p . Let $\mu \in L^2(S, d\sigma)$, $\|\mu\|_2 = 1$ be supported by a neighborhood of p on S . For R large, define $(\hat{\mu})^*$ on $B_R \cap H$ as

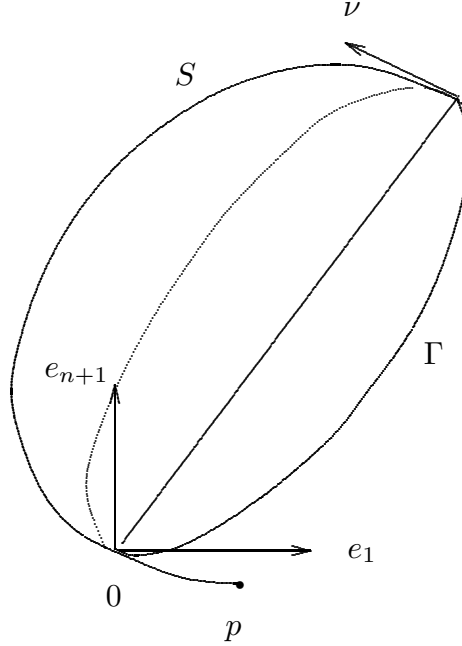
$$(\hat{\mu})^*(x') = \sup_{|t| < R} |\hat{\mu}(x' + t\tau)| \quad (5.5)$$

and estimate

$$\|(\hat{\mu})^*\|_{L^2(H \cap B_R)}. \quad (5.6)$$

Our aim is to describe some examples for which (5.6) is even larger than $R^{\frac{1}{2} - \frac{1}{n}}$ as obtained in Theorem 2, hence setting a further limitation to what one may hope to prove in this general setting.

Consider in the hyperplane $V = [z_1 = \varepsilon z_{n+1}]$ a smooth $(n-1)$ -dim oval Γ with positive curvature, such that $0 \in \Gamma$ and $[e_2, \dots, e_n]$ is the tangent hyperplane of Γ at 0 . We may clearly produce a hypersurface S in \mathbb{R}^{n+1} as above, such that $S \cap V = \Gamma$ and moreover the normal $\nu = \frac{1}{\sqrt{1+\varepsilon^2}}(-e_1 + \varepsilon e_{n+1})$ of V is tangent to S at each point of Γ . Hence, the normal of S at 0 equals $-\frac{1}{\sqrt{1+\varepsilon^2}}(\varepsilon e_1 + e_{n+1})$ and for ε small enough, the point $p \in S$ with normal $\tau = -e_{n+1}$ will be close to 0 . Thus $H = [e_2, \dots, e_n]$



Let $\mathcal{E}_0 = R^{-\frac{1}{2n}} \mathcal{L} \cap \Gamma_0$ with Γ_0 the intersection of Γ with a fixed neighborhood of 0 and \mathcal{L} a suitable regular lattice in V of volume $O(1)$.

Proceeding as in §4, the lattice \mathcal{L} can be chosen s.t.

$$\pi_{x'}(\mathcal{L}^*) + [-CR^{\frac{n-1}{2n}}, CR^{\frac{n-1}{2n}}]e_1 \quad (5.7)$$

is $O(R^{-\frac{1}{2n}})$ dense in $[e_1, \dots, e_n]$, hence

$$\pi_{x'}(R^{\frac{1}{2n}}\mathcal{L}^* + [-CR^{\frac{1}{2}}, CR^{\frac{1}{2}}]\nu) \quad (5.8)$$

is $o(1)$ -dense on $[e_1, \dots, e_n]$.

Note that for $n = 2$, a suitable choice of the curve Γ permits to ensure that

$$|\mathcal{E}_0| = |\mathcal{L} \cap R^{\frac{1}{4}}\Gamma_0| \sim R^{\frac{1}{8}}. \quad (5.9)$$

For $n \geq 3$, taking Γ to be an $(n-1)$ -sphere, one may for appropriate R obtain

$$|\mathcal{E}_0| = |\mathcal{L} \cap R^{\frac{1}{2n}}\Gamma_0| \sim R^{\frac{n-2}{2n}}. \quad (5.10)$$

Let $\rho > 0$ be a small constant. Recalling that ν is tangent to S at the points of Γ , each point from the set $D = \mathcal{E}_0 + [0, \frac{\rho}{\sqrt{R}}]\nu$ is at distance at most $0(\frac{\rho^2}{R})$ from S . This clearly allows to construct a measure μ on S , $\frac{d\mu}{d\sigma} \in L^2(S, \sigma)$, $\|\frac{d\mu}{d\sigma}\|_2 = 1$, such that for $|x| < R$,

$$\hat{\mu}(x) \sim \left(\frac{\rho}{\sqrt{R}}\right)^{-\frac{1}{2}} \left(\frac{\rho}{R}\right)^{\frac{n-1}{2}} |\mathcal{E}_0|^{-\frac{1}{2}} \sum_{\xi \in \mathcal{E}_0} \left[e(x, \xi) \left(\int_0^{\frac{\rho}{\sqrt{R}}} e(sx \cdot \nu) ds \right) + O\left(\frac{\rho^2}{\sqrt{R}}\right) \right]. \quad (5.11)$$

If $x = y + \lambda\nu$ with $y \in R^{\frac{1}{2n}}\mathcal{L}^* \subset V$ and $|\lambda| < CR^{\frac{1}{2}}$, we have $\|x \cdot \xi\| = \|y \cdot \xi\| = 0$ ($\|\cdot\|$ referring to the distance to the nearest integer) by definition of \mathcal{E}_0 , while $\|sx \cdot \nu\| \leq \frac{\rho}{\sqrt{R}}|\lambda| < O(\rho)$. Hence

$$(5.11) \sim R^{-\frac{n-1}{2} - \frac{1}{4}} |\mathcal{E}_0|^{\frac{1}{2}} \quad (5.12)$$

holds for x in an $o(1)$ -neighborhood of $R^{\frac{1}{2n}}\mathcal{L}^* + [-CR^{\frac{1}{2}}, CR^{\frac{1}{2}}]\nu$.

Recalling (5.8), the preceding implies that μ introduced above satisfies

$$\sup_{|x_{n+1}| \lesssim R} |\hat{\mu}(x', x_{n+1})| \gtrsim R^{-\frac{n}{2} + \frac{1}{4}} |\mathcal{E}_0|^{\frac{1}{2}} \quad (5.13)$$

for $x' \in B_R \cap \mathbb{R}^n$, and therefore

$$\|(\hat{\mu})^*\|_{L^2_{[|x'| \leq R]}} \gtrsim R^{\frac{1}{4}} |\mathcal{E}_0|^{\frac{1}{2}}. \quad (5.14)$$

For $n = 2$, (5.9) gives a lower bound $R^{\frac{5}{16}}$ while for $n \geq 3$, we obtain $R^{\frac{1}{2} - \frac{1}{2n}}$.

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